

# Ornstein-Uhlenbeck semi-groups on stratified groups

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## Abstract

We consider, in the setting of stratified groups  $G$ , two analogues of the Ornstein-Uhlenbeck semi-group, namely Markovian diffusion semi-groups acting on  $L^q(p(\gamma)d\gamma)$ , whose invariant density  $p$  is a heat kernel at time 1 on  $G$ .

The first one is symmetric on  $L^2(pd\gamma)$ , its generator is  $\sum_{i=1}^n X_i^* X_i$ , where  $(X_i)_{i=1}^n$  is a basis of the first layer of the Lie algebra of  $G$ .

The second one, denoted by  $T_t = e^{-tN}$ ,  $t > 0$ , is non symmetric on  $L^2(pd\gamma)$  and the formal real part of  $N$  is  $\sum_{i=1}^n X_i^* X_i$ . The operators  $e^{-tN}$  are compact on  $L^q(pd\gamma)$ ,  $1 < q < \infty$ . The spectrum of  $N$  on this space is the set of integers  $\mathbb{N}$  if polynomials are dense in  $L^2(p(\gamma)d\gamma)$ , i.e if  $G$  has at most 4 layers; and we determine in this case its eigenspaces. When  $G$  is step 2, we give another description of these eigenspaces, very similar to the classical definition of "Hermite polynomials" by their generating function.

*Keywords:* stratified groups, sub Laplacian, heat kernel measure, Ornstein-Uhlenbeck semi-groups.

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## 1 Introduction and notation

Let  $G$  be a stratified Lie group equipped with its (biinvariant) Haar measure  $dg$  and dilations  $(\delta_t)_{t \geq 0}$ . Let  $Q$  be the homogeneous dimension of  $G$ . We denote by  $\mathcal{D}(G)$  the space of  $\mathcal{C}^\infty$  compactly supported functions on  $G$ , by  $\mathcal{S}(G)$  the space of Schwartz functions, by  $\mathcal{S}'(G)$  its dual, and  $L^q(\varphi dg) = L^q(G, \varphi dg)$  for a measurable non negative function  $\varphi$ .

As usual, elements  $Z$  of the Lie algebra  $\mathcal{G}$  are identified with left invariant vector fields by

$$(Zf)(g) = \frac{d}{dt} \Big|_{t=0} f(g \exp tZ).$$

Let  $L$  be a subLaplacian on  $G$ , i.e. an operator on  $\mathcal{S}(G)$  defined by

$$L = - \sum_1^n X_i^2 \tag{1}$$

where  $(X_i)_{1 \leq i \leq n}$  is a linear basis of the first layer of  $\mathcal{G}$ . Obviously  $L$  commutes with left translations and satisfies

$$\delta_{t^{-1}} L \delta_t = t^2 L, \quad t > 0. \tag{2}$$

The following facts are well known, see e.g. [FS, propositions 1.68, 1.70, 1.74]:  $-\frac{L}{2}$  generates a strongly continuous semi-group  $e^{-\frac{t}{2}L}$  of convolution operators which are contractions on  $L^q(dg)$ ,  $1 \leq q \leq \infty$ . The kernel  $p_t$  of  $e^{-\frac{t}{2}L}$  is a positive function such that  $p_t(g) = p_t(g^{-1})$ , it lies in  $\mathcal{S}(G)$  and has norm one in  $L^1(dg)$ . Denoting  $p_1 = p$ ,

$$p_t(g) = t^{-\frac{Q}{2}} p \circ \delta_{\frac{1}{\sqrt{t}}}(g).$$

Equivalently, for  $f \in L^q(dg)$ ,

$$e^{-\frac{t}{2}L}(f)(\gamma) = f * p_t(\gamma) = \int_G f(\gamma g^{-1}) p_t(g) dg = \int_G f(\gamma \delta_{\sqrt{t}} g^{-1}) p(g) dg. \tag{3}$$

The aim of this paper is to generalize the Ornstein-Uhlenbeck semi-group in the setting of stratified groups, namely to consider Markovian semi-groups acting on  $L^q(p(\gamma)d\gamma)$ ,  $1 \leq q \leq \infty$ , for which  $p(\gamma)d\gamma$  is an invariant measure, whose generators are related to the first layer gradient

$$\nabla = (X_1, \dots, X_n).$$

The classical Ornstein-Uhlenbeck semi-group is defined on  $\mathcal{S}(\mathbb{R}^n)$  by Mehler formula

$$e^{-tN_0}(f)(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) p(y) dy, \quad t \geq 0,$$

where the gaussian density  $p(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|y|^2}$  is the kernel of  $e^{-\frac{\Delta}{2}}$ , and  $\Delta$  is the (positive) Laplacian on  $\mathbb{R}^n$ . The O-U semi-group is contracting on  $L^q(\mathbb{R}^n, pdx)$ ,  $1 \leq q \leq \infty$ , compact if  $1 < q < \infty$ , but not compact on  $L^1(\mathbb{R}, pdx)$  [D, theorem 4.3.5], and  $p$  is an invariant measure. The generator  $-N_0$  satisfies

$$N_0 = \sum_{j=1}^n \left( \frac{\partial}{\partial x_j} \right)^* \frac{\partial}{\partial x_j} = \Delta - \sum_{j=1}^n \frac{\frac{\partial p}{\partial x_j}}{p} \frac{\partial}{\partial x_j} = \Delta + \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} = \Delta + A$$

where  $(\frac{\partial}{\partial x_j})^*$  denotes the adjoint on  $L^2(\mathbb{R}^n, pdx)$  and  $A$  is the generator of dilations on  $\mathbb{R}^n$ . On  $L^q(\mathbb{R}^n, pdx)$ ,  $1 < q < \infty$ , the spectrum of  $N_0$  is  $\mathbb{N}$ , and the Hermite polynomials on  $\mathbb{R}^n$  form an orthogonal basis of eigenvectors of  $e^{-tN_0}$  in  $L^2(\mathbb{R}^n, pdx)$ .

The generator  $N_0$  has a fruitful generalization in (commutative or non commutative) analysis on deformed or  $q$ -Fock spaces, namely the number operator  $N$ , i.e. the second differential quantization of identity. A substitute of Mehler formula holds and  $(e^{-tN})_{t>0}$  is the compression of a one parameter group of unitary dilations, see e.g. [LP<sub>2</sub>].

Our motivation in this paper is to exploit Mehler formula in another direction: in the setting of stratified groups Mehler formula still defines a semi-group  $(e^{-tN})_{t>0}$  and we study which properties of the classical O-U semi-group remain valid. We also hope that this semi-group might throw some light on properties of the heat density  $p$ .

### Results and organization of the paper

In section 2 we recall some properties of the self-adjoint semi-group on  $L^2(pd\gamma)$  whose generator is  $-\nabla^* \nabla = -\sum_{i=1}^n X_i^* X_i$ ,  $X_i^*$  being the formal adjoint of  $X_i$  with respect to  $L^2(pd\gamma)$ . We give in passing a simple proof of the known Poincaré inequality in  $L^2(pd\gamma)$ .

In the main section 3 we consider another generalization, the Mehler semi-group, which is defined for  $t \geq 0$  by (theorem 3)

$$T_t(f)(\gamma) = \int_G f(\delta_{e^{-t}} \gamma \delta_{\sqrt{1-e^{-2t}}} g) p(g) dg = e^{-tN}(f)(\gamma).$$

Some properties are described in 3.2, in particular  $pd\gamma$  is an invariant measure. This semi-group is not selfadjoint on  $L^2(pd\gamma)$ , but formally the real part of its generator  $-N$  is  $-\nabla^* \nabla$  and  $N = L + A$  where  $A$  is the generator of the group  $(\delta_{e^t})_{t \in \mathbb{R}}$  of dilations, studied in 3.3.

We show in 3.4 that every  $T_t, t > 0$ , is compact on  $L^q(pd\gamma), 1 < q < \infty$ , (proposition 6), with common spectrum  $e^{-t\mathbb{N}}$  on the closed subspace spanned by polynomials (theorem 7), which coincides with the whole space only if the number of layers of  $\mathcal{G}$  is  $\leq 4$  (proposition 8). We describe the eigenspaces in this case.

In 3.5 we give another description of these eigenspaces if  $G$  is step two, similar to the usual definition of one variable Hermite polynomials by their generating function.

**More notation**

We denote  $\mathcal{G} = V_1 \oplus \dots \oplus V_k$ , where  $V_1, \dots, V_k$  are the layers of the Lie algebra  $\mathcal{G}$  of  $G$ ,  $V_k = \mathcal{Z}$  being the central layer, so that [FS, p. 5]

$$[V_j, V_h] \subset V_{j+h}, \quad [V_1, V_h] = V_{h+1}, \quad 1 \leq h < k$$

The homogeneous dimension of  $G$  is

$$Q = \sum_{j=1}^k j \dim V_j.$$

Generic elements of the layers are denoted respectively by  $X, Y, \dots, U$ , and respective basis of the layers are denoted by  $(X_1, \dots, X_n), (Y_1, \dots, Y_m), \dots, (U_1, \dots, U_k)$ . Such a basis is also denoted by  $(Z_j)_{1 \leq j \leq N}$ . We denote accordingly

$$\begin{aligned} g &= \exp\left(\sum x_i X_i + \sum y_i Y_i + \dots + \sum u_i U_i\right) = \exp(X + Y + \dots + U) \\ &= (x, y, \dots, u) = \exp\left(\sum_{j=1}^N z_j Z_j\right) = (z_j)_{j=1}^N, \end{aligned}$$

since the mapping  $(z_j)_{j=1}^N \rightarrow g$  is a diffeomorphism:  $\mathbb{R}^N \rightarrow G$ .

We denote by  $\mathcal{P}$  the space of polynomials on  $G$ , as defined in [FS, chapter I-C] for the fixed basis  $(Z_j)_{j=1}^N$ : they are polynomials w.r. to the coordinates  $z_j, 1 \leq j \leq N$ .

The dilation  $\delta_t, t \geq 0$ , are defined on  $\mathcal{G}$  and  $G$  by

$$\delta_t(X + Y + \dots + U) = tX + t^2Y + \dots + t^kU, \quad \delta_t(\exp Z) = \exp \delta_t(Z), \quad Z \in \mathcal{G}.$$

For a function  $f$  on  $G$ ,

$$\delta_t(f) = f \circ \delta_t.$$

The generator  $A$  of the one parameter group  $(\delta_{e^s})_{s \in \mathbb{R}}$  of dilations on  $G$  satisfies: for  $f \in \mathcal{S}(G)$  and  $s > 0$

$$\frac{d}{dt} \big|_{t=1} f \circ \delta_t = A(f) = -t t^A \frac{d}{dt} t^{-A}(f) = -t \delta_t \frac{d}{dt} (f \circ \delta_{\frac{1}{t}}). \quad (4)$$

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## 2 The semi-group $e^{-t\nabla^*\nabla}$ on $L^2(pdg)$

This semi-group has already been introduced in [BHT], under a probabilistic point of view, in connection with some Markov processes on Lie groups. We use instead an analytic point of view as in [O]. We consider this semi-group firstly because it is a natural generalization of the classical O-U semi-group, secondly because its generator  $\nabla^*\nabla$  is the real part of the generator  $N$  we shall study in part 3, see theorem 3.

### 2.1 Definition and some properties

We consider the (closed) accretive sesquilinear form

$$a(f, h) = \int_G (\nabla f \cdot \nabla h) pdg = \int_G \sum_{i=1}^n X_i f \overline{X_i h} pdg$$

whose (dense) domain in  $L^2(pdg)$  is the Hilbert space

$$H^1(p) = \{f \in L^2(pdg) \mid X_i f \in L^2(pdg), 1 \leq i \leq n\}$$

equipped with the norm  $\|f\|_{H^1(p)}^2 = \|f\|_{L^2(p)}^2 + \|\nabla f\|_{L^2(p)}^2$ ; this form is continuous on  $H^1(p) \times H^1(p)$ .

Hence [O, proposition 1.51, theorem 1.53] it defines an operator, which we denote by  $\nabla^*\nabla$ , such that  $-\nabla^*\nabla$  is the generator of a strongly continuous semi-group of contractions on  $L^2(pdg)$ ; moreover this semi-group is holomorphic on the sector  $\Sigma_{\frac{\pi}{2}} = \{|\arg z| < \frac{\pi}{2}, z \neq 0\}$ , and  $e^{-z\nabla^*\nabla}$  is a contraction on  $L^2(pdg)$  for  $z \in \Sigma_{\frac{\pi}{2}}$ . Obviously, on  $\mathcal{S}(G)$ ,

$$\nabla^* \nabla = \sum_{i=1}^n X_i^* X_i = L - \sum_{i=1}^n \frac{X_i p}{p} X_i = L - B. \quad (5)$$

Since  $X_i$  is a derivation, the chain rule holds, hence  $X_i(f^+) = (X_i f)1_{\{f>0\}}$  by the same proof as for usual derivations on  $\mathbb{R}^N$  [O, proposition 4.4], and  $a(f^+, f^-) = 0$ ; since the form  $a$  also preserves real valued functions, the semi-group  $e^{-t\nabla^* \nabla}$  is positivity preserving [O, theorem 2.6]. Since  $e^{-t\nabla^* \nabla}(1) = 1$ , the semi-group is thus contracting on  $L^\infty(pdg)$ . Since moreover  $\nabla^* \nabla$  is self-adjoint,  $e^{-t\nabla^* \nabla}$  is measure preserving, i.e.

$$\int_G e^{-t\nabla^* \nabla}(f) pdg = \int_G f pdg, t > 0,$$

so it extends as a contraction semi-group on  $L^1(pdg)$  hence on  $L^q(pdg)$ ,  $1 < q < \infty$  by interpolation.

## 2.2 Poincaré inequality in $L^2(pdg)$

Poincaré inequality [DM, theorem 4.2] means that the spectrum of  $\nabla^* \nabla$  on  $L^2(pdg)$  lies in  $\{0\} \cup [C^{-1}, \infty[$ : there exists  $C > 0$  such that, for  $f \in \mathcal{S}(G)$ ,

$$\left\| f - \int_G f pdg \right\|_{L^2(pdg)}^2 \leq C \int_G |\nabla f|^2 pdg = C \int_G f(\nabla^* \nabla f) pdg. \quad (6)$$

(6) follows from the inequality (used for  $q = 2$ ) [DM, theorem 4.1]

$$|\nabla(e^{-tL} f)|^q \leq C_q e^{-tL} (|\nabla f|^q), \quad 1 < q < \infty, \quad (7)$$

which B. Driver and T. Melcher proved, first for  $\mathbb{H}_1$ , then for nilpotent groups  $G$  (see T. Melcher's thesis), using Malliavin calculus. See also [BHT] for some extensions.

We shall show in proposition 1 that (7) also follows easily from gaussian estimates of  $p$  and  $\nabla p$ .

Using the explicit formula for the Carnot-Caratheodory distance, H.Q. Li [Li, corollary 1.2] obtained (7) for  $q = 1$ , on the 3-dimensional Heisenberg group  $G = \mathbb{H}_1$ . As well known [A, théorème 5.4.7], this implies Log-Sobolev inequality for the measure  $pdg$  on  $\mathbb{H}_1$  and (6). Another proof of this Log-Sobolev inequality for  $\mathbb{H}_1$ , hence for  $\mathbb{H}_k$ , is given in [HZ, theorem 7.3].

**Proposition 1** [DM] *Let  $G$  be a stratified group. Then (7) and Poincaré inequality (6) hold true.*

Proof: By [DM, theorem 4.2, proposition 2.6, lemma 2.3] it is enough to prove (7) for  $t = \frac{1}{2}$ , at  $\gamma = 0$ . Hence, it is enough to prove, for an element  $X$  of the basis of  $V_1$ , and  $f \in \mathcal{S}(G)$ ,

$$\left| X(e^{-\frac{1}{2}L}f)(0) \right| = |X(f * p)(0)| = \left| \int_G (\widehat{X}f)(g)p(g)dg \right| \leq C_{q,X} \|\nabla f\|_{L^q(pdg)};$$

here [FS, p. 22 and proposition 1.29]

$$(\widehat{X}f)(g) = \frac{d}{dt} \big|_{t=0} f((\exp tX)g), \quad \widehat{X} = X + \sum_{j>n} Q_{X,j}Z_j$$

where  $(Z_j)_{j=1}^N$  is a basis of  $\mathcal{G}$  respecting the layers and  $Q_{X,j}$  is a polynomial (with homogeneous degree  $h-1$  if  $Z_j \in V_h, 2 \leq h \leq k$ ).

Since  $[V_1, V_{h-1}] = V_h, 2 \leq h \leq k$ , we may choose  $Z_j \in V_h$  such that  $Z_j = [Y, A]$ , where  $Y$  is an element of the basis of  $V_1$  and  $A \in V_{h-1}$ . Then

$$\left| \int_G Z_j f(g) Q_{X,j}(g) p(g) dg \right| \leq \left| \int_G Y f A(Q_{X,j}p) dg \right| + \left| \int_G A f Y(Q_{X,j}p) dg \right|.$$

Iterating for  $A \in V_1 + \dots + V_{k-1}$  and so on,  $\left| \int_G (\widehat{X}f)(g)p(g)dg \right|$  is finally less than a finite number (which does not depend on  $f$ ) of terms  $\left| \int_G Y f Z(Qp) dg \right|$  where  $Y$  is an element of the basis of  $V_1$ ,  $Z \in \mathcal{G}$ , and  $Q$  is a polynomial. Each of these terms can be estimated by

$$\left| \int_G Y f Z(Qp) dg \right| \leq \|\nabla f\|_{L^q(pdg)} (\|ZQ\|_{L^{q'}(pdg)} + \left\| Q \frac{Zp}{p} \right\|_{L^{q'}(pdg)})$$

where  $\frac{1}{q} + \frac{1}{q'} = 1$ . Then  $\|ZQ\|_{L^{q'}(pdg)}$  is finite since  $ZQ$  is a polynomial and  $p \in \mathcal{S}(G)$ . The main point is that  $\left\| Q \frac{Zp}{p} \right\|_{L^{q'}(pdg)}$  is finite. Indeed, denoting  $d(g) = d(0, g)$  where  $d$  is the Carnot-Caratheodory distance on  $G$ , one uses [CSV, theorem IV.4.2 and Comments on chapter IV]: for  $0 < \varepsilon < 1$ ,

$$C_\varepsilon e^{-\frac{1}{2-2\varepsilon}d^2(g)} \leq p(g) \leq K_\varepsilon e^{-\frac{1}{2+2\varepsilon}d^2(g)}. \quad (8)$$

and, for  $Z \in \mathcal{G}$ ,

$$(Zp)(g) \leq K_{\varepsilon,Z} e^{-\frac{1}{2+2\varepsilon}d^2(g)}. \quad (9)$$

Hence  $Q \frac{Zp}{p}$  lies in  $L^r(pdg), 1 \leq r < \infty$ , which ends the proof. ■

### 3 Definition and properties of the Mehler semi-group

#### 3.1 Preliminaries

The next proposition extends a classical property of independant gaussian variables and will imply the semi-group property of our family of operators.

**Proposition 2** *Let  $\gamma, g$  be independant  $G$ -valued random variables with law  $pdg$ . Then the r.v.*

$$\delta_{\cos \theta} \gamma \delta_{\sin \theta} g, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

*has the same law, i.e. for any bounded borelian function  $f$  on  $G$ ,*

$$\int_{G^2} f(\delta_{\cos \theta} \gamma \delta_{\sin \theta} g) p(\gamma) p(g) d\gamma dg = \int_G f(g) p(g) dg.$$

*More generally, if  $g_1, \dots, g_n$  are  $G$ -valued i.i.d r.v. with law  $pdg$  and  $\sum_{1 \leq j \leq n} a_j^2 = 1, (a_j \geq 0)$ , the law of  $\prod_{j=1}^{j=n} \delta_{a_j} g_j$  is  $pdg$ .*

Proof: By two changes of variables, denoting  $C = \sin \theta \cos \theta$ ,

$$\begin{aligned} \int_{G^2} f(\delta_{\cos \theta} \gamma \delta_{\sin \theta} g) p(g) p(\gamma) d\gamma dg &= \frac{1}{CQ} \int_{G^2} f(\gamma' g') p(\delta_{\frac{1}{\cos \theta}} \gamma') p(\delta_{\frac{1}{\sin \theta}} g') d\gamma' dg' \\ &= \frac{1}{CQ} \int_{G^2} f(g) p(\delta_{\frac{1}{\cos \theta}} \gamma') p(\delta_{\frac{1}{\sin \theta}} (\gamma'^{-1} g)) d\gamma' dg \\ &= \int_G f(g) (p_{\cos^2 \theta} * p_{\sin^2 \theta})(g) dg \\ &= \int_G f(g) p(g) dg. \end{aligned}$$

The second assertion follows by iteration.

*Remark 1:* A central limit theorem for i.i.d centered random variables with values in a stratified group  $G$  and law  $\mu$  with order 2 moments is proved in [CR, theorem 3.1]. The density  $p$  of the limit law is the kernel at time 1 of a diffusion semi-group whose generator satisfies (2).

*Remark 2:* If  $X, Y$  are i.i.d standard gaussian vectors with values in  $\mathbb{R}^n$ , the couple  $(X \cos \theta + Y \sin \theta, \frac{d}{d\theta}(X \cos \theta + Y \sin \theta))$  has the same joint law as



$(X, Y)$ . This fact implies, in the O-U case, that  $\cos^{N_0} \theta$  is the compression of the isometry  $R_\theta$  of  $L^2(\mathbb{R}^n \times \mathbb{R}^n, p(x)p(y)dxdy)$  defined by

$$R_\theta(F)(x, y) = F(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$$

and  $(R_\theta)_{\theta \in \mathbb{R}}$  is a one parameter group preserving the measure  $p(x)p(y)dxdy$ . This point of view was exploited e.g. in [P, theorem 2.2] in order to get a concentration inequality for the gaussian measure .

In the stratified setting we were not able to exhibit explicit unitary dilations for the Mehler operators  $T_t$  defined below.

### 3.2 The Mehler semi-group

We now define the Mehler semi-group on  $L^q(G, pdg)$ .

**Theorem 3** *Let  $L$ , defined by (1), be a subLaplacian on a stratified group  $G$ , and let  $p$  be the kernel of  $e^{-\frac{L}{2}}$ .*

*a) The family of operators  $(T_t)_{t \geq 0}$  defined on  $\mathcal{S}(G)$  by*

$$T_t(f)(\gamma) = \int_G f(\delta_{e^{-t}\gamma} \delta_{\sqrt{1-e^{-2t}}} g) p(g) dg = e^{-\frac{L}{2}(1-e^{-2t})}(f)(\delta_{e^{-t}}\gamma) \quad (10)$$

*is a semi-group whose generator  $-N$  is defined on  $\mathcal{S}(G)$  by*

$$N = L + A. \quad (11)$$

*b) The probability measure  $pd\gamma$  is invariant by  $(T_t)_{t \geq 0}$  i.e.*

$$\int_G T_t(f)(\gamma) p(\gamma) d\gamma = \int_G f(\gamma) p(\gamma) d\gamma \quad (12)$$

*and, for  $f \in \mathcal{S}(G)$ ,  $\int_G (Nf) pdg = 0$ .*

*c)  $(T_t)_{t \geq 0}$  extends as a Markovian semi-group of contractions on  $L^q(G, pd\gamma)$ ,  $1 \leq q \leq \infty$ , strongly continuous if  $q \neq \infty$ .*

*d) If  $f \in L^q(pd\gamma)$ ,  $1 \leq q < \infty$ ,*

$$\left\| T_t(f) - \int_G f pdg \right\|_{L^q(pd\gamma)} \xrightarrow{t \rightarrow \infty} 0.$$

e)  $(T_t)_{t>0}$  is not self-adjoint on  $L^2(G, pd\gamma)$  as soon as  $G$  is not abelian. Formally  $\nabla^* \nabla$  is the real part of  $N$ , i.e., for  $f, h \in \mathcal{S}(G)$ ,

$$\langle Nf, h \rangle_{L^2(p)} = \langle (\nabla^* \nabla + iC)f, h \rangle_{L^2(p)}$$

where  $C$  is a non zero first order differential operator satisfying  $\langle Cf, h \rangle = \langle f, Ch \rangle$ . In particular, for  $f \in \mathcal{S}(G)$ ,

$$\Re \int_G (Nf) f pd\gamma = \int_G |\nabla f|^2 pd\gamma = \int_G (\nabla^* \nabla f) f pd\gamma.$$

If moreover  $f$  is real valued, the left integral is real.

By the change of notation  $e^{-t} = \cos \theta$ ,  $< \theta < \frac{\pi}{2}$ , (10) can be rewritten as

$$\cos^N \theta(f)(\gamma) = \int_G f(\delta_{\cos \theta} \gamma \delta_{\sin \theta} g) p(g) dg = \delta_{\cos \theta} \circ e^{-\frac{1}{2} \sin^2 \theta L}(f)(\gamma). \quad (13)$$

Proof: a) Let  $\varphi(g') = T_t(f)(g')$ ; we compute

$$\begin{aligned} T_s(\varphi)(\gamma) &= \int_G \varphi(\delta_{e^{-s}} \gamma \delta_{\sqrt{1-e^{-2s}}} h) p(h) dh \\ &= \int_{G^2} f(\delta_{e^{-t}} [\delta_{e^{-s}} \gamma \delta_{\sqrt{1-e^{-2s}}} h] \delta_{\sqrt{1-e^{-2t}}} g) p(g) p(h) dg dh \\ &= \int_G f(\delta_{e^{-(t+s)}} \gamma \delta_{\sqrt{1-e^{-2(s+t)}}} k) p(k) dk = T_{s+t}(f)(\gamma) \end{aligned}$$

where the third equality comes from proposition 2 applied to  $(h, g)$ . By the chain rule applied to (10),

$$Nf = -\frac{d}{dt} \big|_{t=0} T_t(f) = Lf + A(f).$$

b) Proposition 2 gives (12). Differentiating (12) at  $t = 0$  for  $f \in \mathcal{S}(G)$  implies

$$\int_G (Nf) pdg = 0.$$

Another proof will be given in Remark 3.

c)  $T_t$  is contracting both on  $L^1(G, pd\gamma)$ , since it is positivity and measure preserving, and on  $L^\infty(G, pd\gamma)$ , since it is positivity preserving and  $T_t(1) = 1$ . Hence  $T_t$  is contracting on  $L^q(G, pd\gamma)$ ,  $1 \leq q \leq \infty$  by interpolation.

Since  $\mathcal{D}(G)$  is norm dense in  $L^q(G)$ , it is norm dense in  $L^q(pd\gamma)$ ,  $1 \leq q < \infty$ : indeed, if  $F \in L^{q'}(pd\gamma)$  ( $\frac{1}{q} + \frac{1}{q'} = 1$ ) and  $\int_G f F pd\gamma = 0$  for every  $f \in \mathcal{D}(G)$ , then  $Fp \in L^{q'}(G)$  hence  $Fp = 0$   $d\gamma$  a.s.. Writing  $e^{-t} = \cos \theta$ , one has, for  $f \in \mathcal{D}(G)$ ,

$$\begin{aligned} \|T_t(f) - f\|_{L^q(pd\gamma)}^q &= \left\| \int_G [f(\delta_{\cos \theta \gamma} \delta_{\sin \theta g}) - f(\gamma)] p(g) dg \right\|_{L^q(pd\gamma)}^q \\ &\leq \int_{G^2} |f(\delta_{\cos \theta \gamma} \delta_{\sin \theta g}) - f(\gamma)|^q p(\gamma) p(g) d\gamma dg, \end{aligned}$$

which converges to 0 as  $\theta \rightarrow 0$  by the dominated convergence theorem. Since  $T_t$  is contracting, the strong continuity on  $L^q(pd\gamma)$  follows by density.

d) Similarly, if  $f$  is bounded and continuous on  $G$ ,

$$f(\delta_{e^{-t}\gamma} \delta_{\sqrt{1-e^{-2t}}g}) \rightarrow_{t \rightarrow \infty} f(g);$$

by dominated convergence theorem  $T_t(f) \rightarrow_{t \rightarrow \infty} \int_G f(g) p(g) dg$  pointwise and in the norm of  $L^q(pd\gamma)$ . The claim follows by density.

e) By (11), (5) and lemma 4 below, for  $f \in \mathcal{S}(G)$ ,

$$(N - \nabla^* \nabla) f = A(f) + \sum_{1 \leq j \leq n} \frac{X_j p}{p} X_j f = \sum_{1 \leq j \leq N} b_j Z_j f$$

where the functions  $b_j$  are not all zero if  $j > n = \dim V_1$ . Hence for  $h \in \mathcal{S}(G)$ ,

$$\int_G (N - \nabla^* \nabla)(f) \bar{h} p dg = - \int_G f \left[ \sum_{1 \leq j \leq N} b_j(g) (Z_j \bar{h}) p + \bar{h} Z_j (b_j p) \right] dg.$$

By b), the left hand side is zero for  $h = 1$ , hence  $\sum_{1 \leq j \leq N} Z_j (b_j p) = 0$ . Since  $T_t$  preserves real valued functions, so does  $N$ , hence

$$\int_G (N - \nabla^* \nabla)(f) \bar{h} p dg = - \int_G f (N - \nabla^* \nabla)(\bar{h}) p dg = - \int_G \overline{f(N - \nabla^* \nabla)(h)} p dg,$$

which proves  $(iC)^* = -iC$ , where  $iC = N - \nabla^* \nabla = A + B$ . The remaining assertions are obvious.

*Remark 3:* We now give another instructive proof of  $\int_G (Nf)pdg = 0$ ,  $f \in \mathcal{S}(G)$ , hence of (12). We claim that, for  $f, h \in \mathcal{S}(G)$ ,

$$\int_G (Nf)hdg = \int_G f[L(h) - Qh + \frac{d}{ds} \big|_{s=1} h \circ \delta_{\frac{1}{s}}]dg = \int_G f(L - QId - A)(h)dg.$$

Indeed,  $N = L + A$ ,  $L$  is formally selfadjoint on  $L^2(dg)$  and the claim follows by differentiating at  $s = 1$  the right hand side of

$$\int_G f(\delta_s \gamma)h(\gamma)d\gamma = s^{-Q} \int_G f(\gamma')h(\delta_{\frac{1}{s}} \gamma)d\gamma'.$$

By (4) and [LP, lemma 2],  $p$  may be precisely defined as the unique solution in  $L^1(G)$ , satisfying  $\int_G p(g)dg = 1$ , of

$$(L - QId - A)(p) = Lp - Qp + s\delta_s \frac{d}{ds}(p \circ \delta_{\frac{1}{s}}) = 0. \blacksquare$$

*Remark 4:* As already mentioned in section 2.2, Log-Sobolev inequality for  $pd\gamma$  is known for  $\mathbb{H}_k$ . It is equivalent both to hypercontractivity of  $e^{-tN}$  and to hypercontractivity of  $e^{-t\nabla^* \nabla}$  on  $\mathbb{H}_k$ , since  $p$  is an invariant measure for these markovian semigroups and  $N, \nabla^* \nabla$  are diffusion operators [A, theorem 2.8.2].

### 3.3 The generator of dilations

We may identify  $G$  with a group of finite matrices [V, theorem 3.6.6]. The derivation formula for an exponential of a matrix valued function, see e.g. [H, theorem 69], applied to a smooth function  $Z(s): \mathbb{R} \rightarrow \mathcal{G}$ , where  $\mathcal{G}$  has  $k$  layers, gives

$$\begin{aligned} \frac{d}{ds} \exp Z(s) &= \lim_{h \rightarrow 0} \frac{\exp Z(s+h) - \exp Z(s)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\exp(Z(s) + hZ'(s)) - \exp Z(s)}{h} \\ &= [\exp Z(s)]V(Z(s)), \end{aligned} \tag{14}$$

where

$$V(Z(s)) = (d \exp)_{Z(s)}(Z'(s)) = Z'(s) + \sum_{l=1}^{k-1} \frac{(-1)^l}{(l+1)!} (AdZ(s))^l(Z'(s)). \quad (15)$$

Hence

$$\exp Z(s+h) = \exp Z(s) \exp h[V(Z(s)) + o(1)],$$

which entails for  $f \in \mathcal{C}^\infty(G)$

$$\frac{d}{ds} f(\exp Z(s)) = V(Z(s))(f)(\exp Z(s)). \quad (16)$$

**Lemma 4** *Let  $A$  be the generator of the group of dilations  $(\delta_{e^t})_{t \in \mathbb{R}}$ . Then*

$$A(f)(g) = \sum_{1 \leq j \leq N} a_j(g) Z_j f(g)$$

where the functions  $a_j$  are polynomials w.r. to the coordinates of  $g$ , and are not all zero for  $j > n = \dim V_1$ .

Proof: Assume that  $\mathcal{G}$  has  $k$  layers,  $k \geq 2$ . Let

$$\delta_s g = \exp(sX + s^2Y + \dots + s^k U) = \exp Z(s).$$

By (16)  $A = V(Z(1))$ . Noting that  $Z' - Z \in V_2 + \dots + V_k$ , we get  $(AdZ(1))^l(Z'(1)) \in V_3 + \dots + V_k, l \geq 1$ . So  $V(Z(1)) - (X + 2Y)$  lies in  $V_3 + \dots + V_k$ . ■

**Notation:** We denote by  $\mathcal{P}_n$  the (finite dimensional) space of homogeneous polynomials on  $G$  with homogeneous degree  $n, n \in \mathbb{N}$ , i.e. satisfying

$$\delta_s(P) = s^n P, \quad P \in \mathcal{P}_n; \quad (17)$$

equivalently,  $\mathcal{P}_n$  is the eigensubspace of  $A$  on  $\mathcal{P}$  associated to  $n$ . The finite dimensional subspaces  $B_n = \mathcal{P}_0 + \dots + \mathcal{P}_n$  are stable under  $L$  and dilations, hence under  $e^{-\frac{tL}{2}}$  and  $\cos^N \theta$  by (10), these operators being naturally extended on  $\mathcal{S}'(G)$ . In particular  $e^{\frac{L}{2}}$  is well defined on  $B_n$  and is the inverse of  $e^{-\frac{L}{2}}$ , which is thus one to one on every  $B_n$  hence on  $\mathcal{P} = \cup_{n \geq 0} B_n$ .

The next lemma is the key for the computation of the spectrum of  $\cos^N \theta$ . It will be exploited again in section 3.5.

**Lemma 5** a) The generator  $A$  of dilations on  $G$  satisfies  $[L, A] = 2L$  on  $\mathcal{C}^\infty(G)$ .

b)  $e^{-\frac{L}{2}} \circ \cos^N \theta = \delta_{\cos \theta} e^{-\frac{L}{2}}$  on  $\mathcal{S}'(G)$ .

c) The set of polynomials  $e^{\frac{L}{2}}(\mathcal{P}_n)$  is a space of eigenvectors of  $\cos^N \theta$  associated to the eigenvalue  $\cos^n \theta, n \geq 0$ .

Proof: a) We rewrite (2) as

$$Le^{tA} = e^{2t} e^{tA} L, \quad t \in \mathbb{R},$$

and a) follows by differentiating at  $t = 0$ .

b) By (3), on  $\mathcal{S}(G)$ , hence on  $\mathcal{S}'(G)$ , for  $t > 0$ ,

$$e^{-\frac{t^2}{2}L} = \delta_{\frac{1}{t}} \circ e^{-\frac{L}{2}} \circ \delta_t. \quad (18)$$

Hence, on  $\mathcal{S}'(G)$ , by (10) and (18) applied to  $t = \cos \theta$ ,

$$e^{-\frac{L}{2}} \circ \cos^N \theta = e^{-\frac{L}{2}} \circ \delta_{\cos \theta} \circ e^{-\frac{\sin^2 \theta}{2}L} = \delta_{\cos \theta} \circ e^{-\frac{L}{2}}.$$

c) Since  $e^{-\frac{L}{2}}$  is invertible on  $\mathcal{P}$ , and  $\mathcal{P}$  is stable under  $\cos^N \theta$ , b) implies on  $\mathcal{P}$

$$\cos^N \theta \circ e^{\frac{L}{2}} = e^{\frac{L}{2}} \circ \delta_{\cos \theta}.$$

Applying this to  $\mathcal{P}_n$  proves the result.

### 3.4 Compacity and spectrum of $\cos^N \theta$ on $L^q(pd\gamma)$

**Proposition 6** Let  $\cos^N \theta$  be defined by (13). Then

a)  $\cos^N \theta$  is a Hilbert-Schmidt operator on  $L^2(pd\gamma)$ .

b)  $\cos^N \theta$  is compact on  $L^q(pd\gamma), 1 < q < \infty$ ; its non zero eigenvalues and corresponding eigenspaces are the same on  $L^2(pd\gamma)$  and  $L^q(pd\gamma)$ . In particular its spectrum  $\sigma(\cos^N \theta)$  does not depend on  $q$  and

$$\sigma(\cos^N \theta) = (\cos \theta)^{\sigma(N)} \cup \{0\}.$$

Actually,  $\cos^N \theta$  is a trace class operator on  $L^2(pd\gamma)$  by a) and the semi-group property of  $(e^{-tN})_{t>0}$ .

Proof: a) We must show that the kernel of  $\cos^N \theta$  lies in  $L^2(G \times G, pd\gamma \otimes pdg)$ . For fixed  $\gamma$  and  $\theta, 0 < \theta < \frac{\pi}{2}$ ,

$$\int_G f(\delta_{\cos \theta} \gamma \delta_{\sin \theta} g) p(g) dg = \frac{1}{\sin^Q \theta} \int_G f(z) p(\delta_{\frac{\cos \theta}{\sin \theta}} \gamma^{-1} \delta_{\frac{1}{\sin \theta}} z) dz,$$

so we must prove the convergence of the integral

$$I(\theta) = \int_{G^2} p^2(\delta_{\frac{\cos \theta}{\sin \theta}} \gamma^{-1} \delta_{\frac{1}{\sin \theta}} z) \frac{p(\gamma)}{p(z)} dz d\gamma.$$

By the gaussian estimates (8)

$$\frac{C_\varepsilon}{K_\varepsilon^3} p^2(\delta_{\frac{\cos \theta}{\sin \theta}} \gamma^{-1} \delta_{\frac{1}{\sin \theta}} z) \frac{p(\gamma)}{p(z)} \leq \exp\left(\frac{d^2(z)}{2-2\varepsilon} - \frac{d^2(\gamma)}{2+2\varepsilon} - \frac{d^2(\delta_{\frac{\cos \theta}{\sin \theta}} \gamma^{-1} \delta_{\frac{1}{\sin \theta}} z)}{1+\varepsilon}\right) = \exp \beta.$$

The Carnot distance  $d$  satisfies

$$d(g) \leq d(\gamma^{-1}g) + d(\gamma) \text{ and } d(\delta_t g) = td(g).$$

Hence

$$\begin{aligned} (1+\varepsilon)\beta &\leq \frac{d^2(z)}{2(1-\varepsilon)^2} - \frac{d^2(\gamma)}{2} - \left(\frac{1}{\sin \theta} d(z) - \frac{\cos \theta}{\sin \theta} d(\gamma)\right)^2 \\ &\leq d^2(z) \left(\frac{1}{2-4\varepsilon} - \frac{1-\cos \theta}{\sin^2 \theta}\right) + d^2(\gamma) \left(\frac{\cos \theta - \cos^2 \theta}{\sin^2 \theta} - \frac{1}{2}\right). \end{aligned}$$

Since  $\frac{1-\cos \theta}{\sin^2 \theta} > \frac{1}{2}$  on  $]0, \frac{\pi}{2}]$ , the coefficient of  $d^2(\gamma)$  is strictly negative, and so is the coefficient of  $d^2(z)$  for small enough  $\varepsilon > 0$ . Hence, for some  $c, C > 0$ ,

$$I(\theta) \leq C \int \int_{G^2} e^{-c(d^2(z)+d^2(\gamma))} dz d\gamma = C \left( \int_G e^{-cd^2(z)} dz \right)^2.$$

By the left hand side of (8), for small  $\varepsilon$ ,

$$C_\varepsilon \int_G e^{-cd^2(z)} dz \leq \int_G p^{2c(1-\varepsilon)}(z) dz,$$

and the last integral is finite since  $p \in \mathcal{S}(G)$ . This proves a).

b) By interpolation, since  $\cos^N \theta$  is compact on  $L^2(p(g)dg)$  and bounded on  $L^\infty(pdg)$  and  $L^1(pdg)$ , it is compact on  $L^q(pdg)$ ,  $1 < q < \infty$ , with the same spectrum and the same eigenspaces associated to non zero eigenvalues [D, theorems 1.6.1 and 1.6.2].

By the compacity on  $L^q(pdg)$ , the set of these eigenvalues is  $\{\cos^\lambda \theta \mid \lambda \in \sigma_q(N)\}$  where  $\sigma_q(N)$  denotes the spectrum of  $N$  on  $L^q(pdg)$  [L, chap. 34.5, theorem 13]. Hence  $\sigma_q(N) = \sigma_2(N)$  is discrete and lies in  $\{\lambda \in \mathbb{C} \mid \Re \lambda \geq 0\}$  since  $\cos^N \theta$  is contracting on  $L^2(pdg)$  (or since  $\Re \langle Nf, f \rangle \geq 0$ ). ■

**Theorem 7** *Let  $G$  be a step  $k$  stratified group.*

1) *If  $k \leq 4$*

a) *the spectrum of  $\cos^N \theta$  on  $L^2(\text{pdg})$  is  $\sigma(\cos^N \theta) = (\cos \theta)^\mathbb{N} \cup \{0\}$  and  $\sigma(N) = \mathbb{N}$ .*

b) *the corresponding eigenspaces  $E_n, n \geq 0$ , (which are not pairwise orthogonal in  $L^2(\text{pdg})$ ) are*

$$E_n = e^{\frac{1}{2}L}(\mathcal{P}_n).$$

2) *If  $k > 4$ , assertions a) b) remain true for the restriction of  $\cos^N \theta$  to the closed subspace  $L^2_{\mathcal{P}}(\text{pdg})$  spanned by polynomials.*

If  $k = 1$  polynomials in  $E_n$  are the Hermite polynomials with degree  $n$ .

Proof: 1) follows from 2) and proposition 8 below.

2) We first define  $E_n$  by  $E_n = e^{\frac{1}{2}L}(\mathcal{P}_n)$ . By lemma 5,  $E_n$  lies in the eigenspace of  $\cos^N \theta$  associated to the eigenvalue  $\cos^n \theta$ . By proposition 6,  $\cos^N \theta$  is compact on  $L^2_{\mathcal{P}}(\text{pdg})$ . The claim then follows from the following facts:

Let  $T : E \rightarrow E$  be a compact operator on an infinite dimensional Banach space  $E$ ; let  $\Lambda$  be a set of eigenvalues of  $T$  and let  $E_\lambda, \lambda \in \Lambda$ , be eigensubspaces whose union is total in  $E$ . Then

a) the spectrum of  $T$  is  $\Lambda \cup \{0\}$

b) for  $\lambda \in \Lambda$ ,  $E_\lambda$  is the whole eigenspace associated to  $\lambda$ .

Indeed, assume that  $T$  has an eigenvalue  $\lambda_0 \notin \Lambda$ . Then  $T - \lambda_0 I$  has a closed range with non zero finite codimension (see e.g. [L, chap. 21.1, theorems 3, 4]). But this range contains the linear span of the  $E_\lambda$ 's,  $\lambda \in \Lambda$ , hence is the whole of  $E$ . This is a contradiction, which proves a).

Let  $\lambda_0 \in \Lambda$ ; since  $E_{\lambda_0}$  is stable under  $T$ ,  $T$  acts on the quotient space  $E/E_{\lambda_0}$  and is still compact. The  $E_\lambda$ 's,  $\lambda \in \Lambda \setminus \{\lambda_0\}$  span a dense subspace of  $E/E_{\lambda_0}$ . Applying a) to  $E/E_{\lambda_0}$ ,  $\lambda_0$  cannot belong to the spectrum of  $T$  on the quotient space, which proves b). ■

The proof of the next proposition is essentially due to W. Hebisch (private communication).

**Proposition 8** *Let  $G$  be a stratified group. Then the polynomials are dense in  $L^2(\text{pdg})$  if and only if  $G$  is step  $k$  with  $k \leq 4$ .*

Proof: 1) We recall that polynomials are dense in  $L^2(\mathbb{R}, e^{-c|x|^\alpha} dx)$  if and only if  $\alpha \geq \frac{1}{2}$ : obviously, this does not depend on  $c$  and is equivalent to the density of polynomials in  $L^2(\mathbb{R}^+, e^{-x^\alpha} dx)$ . If  $0 < \alpha < \frac{1}{2}$ , [PS, Part III, problem 153] produces a non zero bounded function  $g_\alpha$  which is orthogonal to polynomials



in  $L^2(\mathbb{R}^+, e^{-\cos(\alpha\pi)x^\alpha} dx)$ . If  $\alpha \geq \frac{1}{2}$ , the result follows from the trick of [Ham, p 197-198]. Indeed, if  $\psi \in L^2(\mathbb{R}^+, e^{-x^\alpha} dx)$  and  $\alpha \geq \frac{1}{2}$ , the function

$$F(z) = \int_{\mathbb{R}^+} \psi(x) e^{\sqrt{x}z} e^{-x^\alpha} dx = \int_{\mathbb{R}^+} \psi(y^2) e^{yz} e^{-y^{2\alpha}} y dy$$

is bounded and holomorphic on  $\{\Re z < \beta\}$  for some  $\beta > 0$ , by Cauchy-Schwarz inequality.

Expanding  $z \rightarrow e^{\sqrt{x}z}$  in power series, one gets  $F(-z) = -F(z)$  if  $\psi$  is orthogonal to polynomials in  $L^2(\mathbb{R}^+, e^{-x^\alpha} dx)$ . Thus  $F$  extends as a bounded entire function, which must be zero by Liouville theorem since  $F(0) = 0$ . Hence the Fourier transform of  $y \rightarrow \psi(y^2) e^{-y^{2\alpha}} y$  is zero, i.e.  $\psi = 0$  a.s..

2) We identify  $g = \exp Z \in G$  with the coordinates  $(x, y, \dots, w)$  of  $Z$  w.r. to a basis respecting the layers and denote

$$\eta(g) = \sum_{i \leq l} |x_i|^2 + \sum_{i \leq m} |y_i|^{\frac{2}{2}} + \dots + \sum_{i \leq r} |w_i|^{\frac{2}{k}}.$$

Obviously  $\eta(\delta_s g) = s^2 \eta(g)$ , in particular  $\eta(g) = d^2(g) \eta(\delta_{\frac{1}{d(g)}} g)$ ,  $d$  denoting the Carnot distance. Since  $\eta$  is strictly positive and bounded on the  $d$ -unit sphere of  $G$ , there exist constants  $c', C' > 0$  such that

$$c' \eta(g) \leq d^2(g) \leq C' \eta(g).$$

By (8) there exist constants  $c, C > 0$  such that the following embeddings

$$L^2(e^{-C\eta(g)} dg) \rightarrow L^2(pdg) \rightarrow L^2(e^{-c\eta(g)} dg)$$

are continuous, with dense ranges since  $\mathcal{D}(G)$  is dense in the three spaces.

3) The algebraic tensor product

$$\mathcal{E} = \otimes_{i \leq l} L^2(e^{-Cx_i^2} dx_i) \otimes \dots \otimes_{i \leq p} L^2(e^{-C|w_i|^{\frac{2}{k}}} dw_i),$$

is dense in  $L^2(e^{-C\eta(g)} dg)$ . For  $k \leq 4$ , one variable polynomials are dense in every factor of  $\mathcal{E}$  by step 1), hence polynomials are dense in  $L^2(e^{-C\eta(g)} dg)$  and in  $L^2(pdg)$ .

Let  $k \geq 5$ . By 1) there exists a non zero function  $g \in L^2(e^{-c|w_r|^{\frac{2}{k}}} dw_r)$  which is orthogonal to polynomials w.r. to  $w_r$ . Then  $1 \otimes \dots \otimes 1 \otimes g \in L^2(e^{-c\eta(g)} dg)$  is orthogonal to all polynomials, so polynomials are neither dense in  $L^2(e^{-c\eta(g)} dg)$ , nor in  $L^2(pdg)$ . ■

### 3.5 Generating functions of polynomial eigenvectors of $N$

The usual Hermite polynomials on  $\mathbb{R}$ , denoted by  $H_n, n \in \mathbb{N}$ , are the eigenvectors of the Ornstein-Uhlenbeck operator  $N_0$ , and have the generating function

$$e^{ixt + \frac{1}{2}t^2} = \sum_{n \geq 0} \frac{(it)^n}{n!} H_n(x) = e^{\frac{1}{2}\Delta}(e^{ixt}) = e^{\frac{1}{2}\Delta} \circ \delta_t(e^{ix}),$$

noting that  $x \rightarrow e^{ix}$  is a bounded eigenvector of  $\Delta$ . In particular

$$i^n H_n(x) = \frac{d^n}{dt^n} \Big|_{t=0} e^{\frac{1}{2}\Delta} \circ \delta_t(e^{ix}).$$

We shall verify (proposition 11) that a similar formula gives polynomial eigenvectors of  $N$ . When  $G$  is step two, these vectors are total in  $L^q(pdg), 1 \leq q < \infty$ , see theorem 12 below. More precisely we give in 3.5.1 a technical lemma producing eigenvectors of  $N$  out of eigenvectors of  $L$ . In 3.5.3 we use this lemma when  $\varphi$  is both an eigenvector of  $L$  and a coefficient function of a representation of  $G$  (proposition 11). We shall first gather in 3.5.2 well known facts about these functions.

#### 3.5.1 Candidates for generating functions of eigenvectors of $N$

In the next lemma 9 we state technical assumptions ensuring the validity of the computation of some eigenvectors of  $N$ . Using lemma 5 b), the point is to define " $e^{\frac{L}{2}}\varphi$ " for suitable functions  $\varphi$  : in lemma 5 c), we choose  $\varphi \in \mathcal{P}$ , here we choose eigenvectors of  $L$ .

**Lemma 9** *Let  $G$  be a stratified group and let  $\varphi \in \mathcal{S}'(G) \cap \mathcal{C}^\infty(G)$  be an eigenvector of  $L$  such that  $L\varphi = \lambda\varphi$ . We assume that, for  $n \geq 1$ ,*

- (i)  $\frac{d^n}{dt^n} \Big|_{t=0} \int_G \delta_t(\varphi)(\gamma g^{-1})p(g)dg = \int_G \frac{d^n}{dt^n} \Big|_{t=0} \delta_t(\varphi)(\gamma g^{-1})p(g)dg$
- (ii)  $\frac{d^n}{dt^n} \Big|_{t=0} \delta_t(\varphi)$  is a polynomial on  $G$ .

*Let*

$$f_t = e^{\frac{t^2\lambda}{2}} \delta_t(\varphi), \quad t > 0; \quad h_n = \frac{d^n}{dt^n} \Big|_{t=0} f_t.$$

*Then  $h_n$  is a polynomial on  $G$  and*

$$\cos^N \theta(h_n) = \cos^n \theta h_n.$$

Proof: Since  $\varphi \in \mathcal{C}^\infty(G)$ ,  $t \rightarrow f_t$  is  $\mathcal{C}^\infty$  on  $\mathbb{R}^+$ . By (2)  $L \circ \delta_t(\varphi) = t^2 \lambda \delta_t(\varphi)$ , so that  $\delta_t(\varphi) = e^{-\frac{L}{2}} f_t$ . By lemma 5 b)

$$e^{-\frac{L}{2}} \cos^N \theta(f_t) = \delta_{\cos \theta} e^{-\frac{L}{2}} f_t = \delta_{\cos \theta} \delta_t(\varphi) = \delta_{t \cos \theta}(\varphi) = e^{-\frac{L}{2}} f_{t \cos \theta}. \quad (19)$$

We claim that

$$\frac{d^n}{dt^n} \big|_{t=0} e^{-\frac{L}{2}} \cos^N \theta(f_t) = e^{-\frac{L}{2}} \cos^N \theta \left( \frac{d^n}{dt^n} \big|_{t=0} f_t \right) = e^{-\frac{L}{2}} \cos^N \theta(h_n). \quad (20)$$

In particular, applying (20) with  $\theta = 0$ ,  $\frac{d^n}{dt^n} \big|_{t=0} e^{-\frac{L}{2}}(f_t) = e^{-\frac{L}{2}}(h_n)$ . Hence, by (20) and (19),

$$e^{-\frac{L}{2}} \cos^N \theta(h_n) = \frac{d^n}{dt^n} \big|_{t=0} e^{-\frac{L}{2}} f_{t \cos \theta} = e^{-\frac{L}{2}} \cos^N \theta(h_n). \quad (21)$$

By Leibnitz rule, it is enough to prove the claim for  $\delta_t(\varphi)$  instead of  $f_t$ . By lemma 5 b) we may replace  $e^{-\frac{L}{2}} \cos^N \theta$  in the claim by  $\delta_{\cos \theta} e^{-\frac{L}{2}}$ . The claim now follows from assumption (i).

By Leibnitz rule and assumption (ii),  $h_n$  is a polynomial. So is  $\cos^N \theta(h_n)$  and the result follows from (21) since  $e^{-\frac{L}{2}}$  is one to one on  $\mathcal{P}$ .

*Remark 5:*  $\varphi$  and  $\varphi \circ \delta_\beta$ ,  $\beta > 0$ , give colinear  $h_n$ 's, since

$$\frac{d^n}{dt^n} \big|_{t=0} e^{\frac{1}{2}t^2\beta^2\lambda} \delta_{t\beta}(\varphi) = \beta^n \frac{d^n}{dt^n} \big|_{t=0} e^{\frac{1}{2}t^2\lambda} \delta_t(\varphi) = \beta^n h_n.$$

### 3.5.2 A total set of eigenvectors of $L$ in $L^q(pdg)$ , $1 \leq q < \infty$ .

Let  $\Pi : G \rightarrow B(L^2(\mathbb{R}^k, d\xi))$  be a non trivial unitary irreducible representation of  $G$ . By definition,  $F \in L^2(\mathbb{R}^k)$  is a  $\mathcal{C}^\infty$  vector for  $\Pi$  if the vector valued function:  $g \rightarrow \Pi(g)(F)$  is  $\mathcal{C}^\infty$  on  $G$ . We still denote by  $\Pi$  the associated differential representation, defined for a  $\mathcal{C}^\infty$  vector  $F$  and  $X \in \mathcal{G}$  by

$$X\Pi(g)(F) = \frac{d}{dt} \big|_{t=0} \Pi(g \exp tX)(F) = \Pi(g)\Pi(X)(F), \quad g \in G, \quad (22)$$

and  $\Pi(X^m) = \Pi(X)^m$ , see e.g. [CG, p.227]; by definition,  $\Pi(X^m)(F)$  still lies in  $L^2(\mathbb{R}^k)$  and is still a  $\mathcal{C}^\infty$  vector for  $\Pi$ .

$\Pi$  extends as a representation of the convolution algebra  $M(G)$  by

$$\Pi(\mu) = \int_G \Pi(g) d\mu(g).$$

In particular  $(\Pi(p_t dg))_{t \geq 0}$  is a semigroup of operators on  $L^2(\mathbb{R}^k)$ , whose generator is  $-\Pi(L)$ . Indeed, for a  $\mathcal{C}^\infty$  vector  $F$ , by (22),

$$\begin{aligned} -\frac{d}{dt} \int_G \Pi(g)(F) p_t(g) dg &= \int_G \Pi(g)(F) (L p_t)(g) dg = \int_G L \circ \Pi(g)(F) p_t(g) dg \\ &= \int_G \Pi(g) \circ \Pi(L)(F) p_t(g) dg \xrightarrow{t \rightarrow 0^+} \Pi(L)(F). \end{aligned}$$

Since  $p \in \mathcal{S}(G)$ ,  $\Pi(p dg) = e^{-\frac{1}{2}\Pi(L)}$  is a trace class operator [CG, theorem 4.2.1]; in particular its non zero eigenvalues are  $\{e^{-\frac{1}{2}\lambda}, \lambda \in \sigma_2(\Pi(L))\}$ , where  $\lambda$  runs through the eigenvalues of  $\Pi(L)$  on  $L^2(\mathbb{R}^k)$ . Moreover, for  $F \in L^2(\mathbb{R}^k)$ , the function  $\Pi(p dg)(F)$  is a  $\mathcal{C}^\infty$  vector for  $\Pi$  [CG, theorem A.2.7 p. 241].

Let  $\mathcal{U}$  be a set of non trivial unitary irreducible representations of  $G$  whose equivalence classes support the Plancherel measure for  $G$ . By Kirillov theory, there exists an integer  $k$ , which does not depend on  $\Pi \in \mathcal{U}$ , such that  $\Pi : G \rightarrow B(L^2(\mathbb{R}^k))$ , see more details in 3.5.4 below.

**Proposition 10** *Let  $G$  be a stratified group and let  $\mathcal{F}$  be the set of coefficient functions*

$$\mathcal{F} = \{\varphi^{\Pi, \mu, \mu'} = \langle \Pi(\cdot)(F_\mu), F_{\mu'} \rangle \mid \Pi \in \mathcal{U}, F_\mu, F_{\mu'} \in \mathcal{B}_\Pi\} \subset L^\infty(dg)$$

where  $\mathcal{B}_\Pi$  is an orthogonal basis of  $L^2(\mathbb{R}^k)$  chosen among eigenvectors of  $e^{-\frac{1}{2}\Pi(L)}$ . Then  $\mathcal{F}$ , which lies in  $\mathcal{C}^\infty(G)$ , is a set of eigenvectors of  $L$  which is total in  $L^q(p(g)dg)$ ,  $1 \leq q < \infty$ .

For fixed  $\Pi, \mu$  the functions  $\{\varphi^{\Pi, \mu, \mu'} \mid F_{\mu'} \in \mathcal{B}_\Pi\}$  are independent and belong to the same eigenspace of  $L$ .

Proof: a) For every non trivial unitary irreducible representation  $\Pi$  of  $G$ , since  $\Pi(p dg)(F_\mu) = e^{-\frac{1}{2}\Pi(L)}(F_\mu) = e^{-\frac{1}{2}\lambda_\mu} F_\mu$ ,  $F_\mu$  is a  $\mathcal{C}^\infty$  vector for  $\Pi$ , hence  $\varphi^{\Pi, \mu, \mu'} \in \mathcal{C}^\infty(G)$ ;  $\varphi^{\Pi, \mu, \mu'}$  is an eigenvector of  $L$  with eigenvalue  $\lambda_\mu$  by (22). Since  $\Pi$  is irreducible, the closed invariant subspace

$$\{F \in L^2(\mathbb{R}^k) \mid \forall g \in G \langle \Pi(g)(F_\mu), F \rangle = 0\}$$

is reduced to  $\{0\}$ , which implies the independence of the  $\varphi^{\Pi, \mu, \mu'}$ 's. (In the Heisenberg case, see [T, p. 19, 51]).

b) Let  $\psi \in L^{q'}(pdg)$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ , be orthogonal to  $\mathcal{F}$ , i.e. for  $\Pi \in \mathcal{U}$ ,

$$0 = \int_G \langle \Pi(g)(F_\mu), F_{\mu'} \rangle \psi(g) p(g) dg = \left\langle \left( \int_G \Pi(g) \psi(g) p(g) dg \right) (F_\mu), F_{\mu'} \right\rangle.$$

Equivalently  $\Pi(\psi p) = \widehat{\psi p}(\Pi) = 0$  for  $\Pi \in \mathcal{U}$ . Then Plancherel formula for  $G$  (see e.g. [CG, theorem 4.3.10]) implies that  $\psi p = 0$   $dg$  a.s.. Indeed, this is clear if  $\psi p \in L^2(dg)$ , in particular if  $q' \geq 2$ . In general,  $\psi p \in L^1(dg)$ ,  $\|(\psi p) * p_t - \psi p\|_{L^1(dg)} \rightarrow_{t \rightarrow 0} 0$  and  $(\psi p) * p_t \in L^2(dg)$ ; moreover  $(\psi p) * p_t = 0$  a.s. since, for every  $\Pi \in \mathcal{U}$ ,

$$\Pi((\psi p) * p_t) = \Pi(\psi p) \Pi(p_t) = 0. \blacksquare$$

### 3.5.3 Polynomial eigenvectors of $N$ built from coefficients of representations

We now consider the functions  $e^{\frac{1}{2}t^2\lambda_\mu} \varphi^{\Pi, \mu, \mu'} \circ \delta_t$  as generating functions of polynomial eigenvectors of  $N$ .

**Proposition 11** *Let  $\varphi^{\Pi, \mu, \mu'} = \langle \Pi(\cdot)(F_\mu), F_{\mu'} \rangle \in \mathcal{F}$  be as in proposition 10. For  $n \geq 1$ , let*

$$h_n^{\Pi, \mu, \mu'} = \frac{d^n}{dt^n} \Big|_{t=0} e^{\frac{1}{2}t^2\lambda_\mu} \varphi^{\Pi, \mu, \mu'} \circ \delta_t.$$

*Then  $h_n^{\Pi, \mu, \mu'}$  is a polynomial eigenvector of  $\cos^N \theta$  with eigenvalue  $\cos^n \theta$ .*

Proof: By proposition 10 and lemma 9, it is enough to prove assumptions (i) and (ii) in lemma 9. We claim the existence of a polynomial  $\psi_n$ ,  $n \geq 1$ , which does not depend on  $t$ , such that, for  $0 \leq t \leq 1$  and  $n \geq 0$ ,

$$\left| \frac{d^n}{dt^n} \varphi^{\Pi, \mu, \mu'} \circ \delta_t \right| \leq \psi_n.$$

Since  $g \rightarrow \psi_n(\gamma g^{-1})$  is still a polynomial, it lies in  $L^1(pdg)$  for every  $\gamma \in G$ , and this will prove assumption (i). We now verify the claim.

**Case 1:** The computation of derivatives being easier if  $G$  is step two, we first consider this setting.

By Schur lemma, the restriction of  $\Pi$  to the center  $\exp \mathcal{Z}$  of  $G$  is given by a character  $u \rightarrow e^{i\langle l, u \rangle}$  where  $l$  is some linear form on  $\mathcal{Z}$ , see e.g. [CG, p. 184]. If  $g = (x, u)$  and  $X = \sum_{j=1}^n x_j X_j \in V_1$ ,

$$\varphi^{\Pi, \mu, \mu'}(\delta_t g) = e^{it^2 \langle l, u \rangle} \langle \Pi(\exp tX)(F_\mu), F_{\mu'} \rangle = e^{it^2 \langle l, u \rangle} \Phi_t^{\Pi, \mu, \mu'}(x)$$

and, by (22),

$$\frac{d^m}{dt^m} \Phi_t^{\Pi, \mu, \mu'}(x) = \langle \Pi(\exp tX) \Pi(X)^m(F_\mu), F_{\mu'} \rangle. \quad (23)$$

Since  $\Pi(X)^m(F_\mu)$  lies in  $L^2(\mathbb{R}^k, d\xi)$ ,  $\langle \Pi(X)^m(F_\mu), F_{\mu'} \rangle$  and  $\|\Pi(X)^m(F_\mu)\|_{L^2(d\xi)}$  are polynomials w.r. to  $x$ ,  $\frac{d^m}{dt^m} \big|_{\alpha=0} \delta_t(\varphi^{\Pi, \mu, \mu'})$  is a polynomial w.r. to  $x, u$ , and  $\left| \frac{d^m}{dt^m} e^{it^2 \langle l, u \rangle} \Phi_t^{\Pi, \mu, \mu'}(x) \right|$  is, for  $0 \leq t \leq 1$ , less than a polynomial  $\psi_n$  which does not depend on  $t$ . This proves (i) and (ii) in this case.

**General case:** As in (14) and (15), for  $g = \exp Z = \exp(X + Y + \dots + U)$  and  $t > 0$ , since  $V(\Pi(\delta_t Z)) = \Pi(V(\delta_t Z))$ ,

$$\frac{d}{dt} \varphi^{\Pi, \mu, \mu'}(\delta_t g) = \frac{d}{dt} \langle \exp \Pi(\delta_t Z)(F_\mu), F_{\mu'} \rangle = \langle \Pi(V(\delta_t Z))(F_\mu), \exp -\Pi(\delta_t Z)(F_{\mu'}) \rangle.$$

At  $t = 0$  this reduces to the polynomial  $\langle \Pi(X)(F_\mu), F_{\mu'} \rangle$ . Since  $\Pi(V(\delta_t Z))$  has polynomial coefficients w.r. to  $t$  and the coordinates of  $g$ , so does  $\|\Pi(V(\delta_t Z))(F_\mu)\|_{L^2(d\xi)}$ . Hence there is a polynomial  $\psi_1$  w.r. to the coordinates of  $g$  such that

$$\sup_{0 \leq t \leq 1} \|\Pi(V(\delta_t Z))(F_\mu)\|_{L^2(d\xi)} \leq \psi_1.$$

This proves the claim for  $n = 1$ . Clearly this can be iterated for upper derivatives, which proves (i) and (ii). ■

### 3.5.4 The step two setting: generalized Hermite polynomials

In this case, the key facts are the extension of the explicit functions  $\varphi^{\Pi, \mu, \mu'} \in \mathcal{F}$  as entire functions on the complexification of  $G$  and the explicit expression of  $p$ . Theorem 12 gives another proof of theorem 7 a) in this setting, with another description of the eigenspaces of  $N$  by generating functions.

**Theorem 12** *Let  $G$  be a step two stratified group. Then*

a) every  $\varphi^{\Pi, \mu, \mu'} \in \mathcal{F}$  lies in the closed subspace of  $L^q(pdg)$ ,  $1 \leq q < \infty$ , spanned by constants and the polynomials  $\{h_n^{\Pi, \mu, \mu'}, n \geq 1\}$  defined in proposition 11.

b) The set of generalized Hermite polynomials

$$\cup_{\varphi^{\Pi, \mu, \mu'} \in \mathcal{F}} \{h_n^{\Pi, \mu, \mu'}, n \geq 1\}$$

together with the constants is a set of eigenvectors of  $N$  which is total in  $L^q(pdg)$ ,  $1 \leq q < \infty$ .

c) For fixed  $n \geq 1$ ,  $\cup_{\varphi^{\Pi, \mu, \mu'} \in \mathcal{F}} \{h_n^{\Pi, \mu, \mu'}\}$  spans the eigenspace of  $N$  associated to  $n$  in  $L^q(pdg)$ ,  $1 < q < \infty$ .

In contrast, if  $G$  has more than 4 layers, assertion b) is false by proposition 8, hence a) is false for some  $\varphi^{\Pi, \mu, \mu'} \in \mathcal{F}$ , by proposition 10. If  $G$  has 3 or 4 layers, we do not know if the conclusions of theorem 12 hold true.

Proof of theorem 12: a) implies b) by propositions 10 and 11.

b) implies c) as recalled in the proof of theorem 7.

a) The proof is given in three steps. In step 1 we state two standard sufficient conditions ensuring statement a); in step 2 we verify these conditions when  $G$  is a Heisenberg group; in step 3 we show how the general step 2 case mimicks the Heisenberg case.

**Step 1:** Let  $\varphi^{\Pi, \mu, \mu'} \in \mathcal{F}$  and assume that

(i) for every  $g \in G$ , the function  $t \rightarrow \varphi^{\Pi, \mu, \mu'}(\delta_t g)$  extends as a holomorphic function  $z \rightarrow \varphi_z^{\Pi, \mu, \mu'}(g)$  on  $\mathbb{C}$ .

(ii) for some connected neighborhood  $\Omega$  of the real axis, for every compact  $K \subset \Omega$ , there exists  $w_K \in L^q(pdg)$ ,  $1 \leq q < \infty$ , such that

$$\left| \varphi_z^{\Pi, \mu, \mu'} \right| \leq w_K, \quad z \in K.$$

We claim that  $\varphi^{\Pi, \mu, \mu'} = \varphi$  then lies in the closed subspace of  $L^q(pdg)$  spanned by  $h_n^{\Pi, \mu, \mu'}, n \geq 1$ . Indeed, let  $\psi \in L^{q'}(pdg)$ ,  $\frac{1}{q} + \frac{1}{q'} = 1$ , and let

$$m(t) = \int_G \varphi(\delta_t g) \psi(g) p(g) dg.$$

By the assumptions,  $m$  extends as a holomorphic function on  $\Omega$  and

$$\frac{d^n}{dz^n} m = \int_G \left( \frac{d^n}{dz^n} \varphi_z \right) \psi p dg, \quad n \geq 0.$$

By proposition 10,  $L(\varphi) = \lambda\varphi$  for some  $\lambda = \lambda_\mu$ . Hence  $t \rightarrow e^{\frac{1}{2}t^2\lambda}m(t)$  also extends as a holomorphic function on  $\Omega$  and

$$\frac{d^n}{dz^n} \big|_{z=0} e^{\frac{1}{2}z^2\lambda}m = \int_G \left[ \frac{d^n}{dz^n} \big|_{z=0} e^{\frac{1}{2}z^2\lambda}\varphi_z \right] \psi p dg = \int_G h_n^{\Pi, \mu, \mu'} \psi p dg, \quad n \geq 0.$$

If  $\psi$  is orthogonal to  $\{h_n^{\Pi, \mu, \mu'}, n \geq 0\}$ , these derivatives are zero, hence  $e^{\frac{1}{2}z^2\lambda}m$  is zero on  $\Omega$ . In particular  $m(1) = 0$ , i.e.  $\psi$  is orthogonal to  $\varphi$ , which proves the claim.

**Step 2: The Heisenberg groups  $\mathbb{H}_k$**

A basis of the first layer of the Lie algebra is  $X_1, Y_1, \dots, X_k, Y_k$  where  $[X_j, Y_j] = -4U$ ,  $U$  spans the center, and the other commutators are zero. By the Campbell-Hausdorff formula,

$$g = \exp\left(\sum_{j=1}^k x_j X_j + y_j Y_j + uU\right) = \exp uU \prod_{j=1}^k \exp(-2x_j y_j U) \exp y_j Y_j \exp x_j X_j.$$

We first consider the Schrödinger (unitary irreducible) representation  $\Pi_S : \mathbb{H}_k \rightarrow B(L^2(\mathbb{R}^k))$ , defined on the Lie algebra by

$$\Pi_S(X_j) = \frac{\partial}{\partial \xi_j}, \quad \Pi_S(Y_j) = i\xi_j, \quad \Pi_S(U) = -\frac{1}{4}\left[\frac{\partial}{\partial \xi_j}, i\xi_j\right] = -\frac{i}{4}I.$$

For  $F \in L^2(\mathbb{R}^k)$ , this implies

$$\Pi_S(g)(F)(\xi) = e^{-i\frac{u}{4}} e^{\frac{i}{2}\sum_{j=1}^k x_j y_j} e^{i\sum_{j=1}^k y_j \xi_j} F(\xi + x), \quad (24)$$

and

$$\Pi_S(L) = H = \sum_{j=1}^k \left(-\frac{\partial^2}{\partial \xi_j^2} + \xi_j^2\right)$$

is the harmonic oscillator. If  $k = 1$ , an o.n. basis of eigenvectors of  $H$  in  $L^2(\mathbb{R})$  is the sequence of Hermite functions  $F_\mu, \mu \in \mathbb{N}$ . The so called special Hermite functions [T, p. 18-19] are, for  $\mu, \mu' \in \mathbb{N}$  and  $\varepsilon_{\mu, \mu'} = \text{sgn}(\mu' - \mu)$ ,

$$\begin{aligned} \langle \Pi_S(x, y, 0)(F_\mu), F_{\mu'} \rangle &= \Phi_{\mu, \mu'}(x, y) = \int_{\mathbb{R}} e^{iy\xi} F_\mu\left(\xi + \frac{x}{2}\right) F_{\mu'}\left(\xi - \frac{x}{2}\right) d\xi \\ &= r_{\mu, \mu'}(x^2 + y^2) e^{-\frac{1}{2}(x^2 + y^2)} (x + i\varepsilon_{\mu, \mu'} y)^{|\mu - \mu'|}, \quad (25) \end{aligned}$$



where  $r_{\mu,\mu'} = r_{\mu',\mu}$  is a one variable polynomial with real coefficients.

An o.n basis of eigenvectors of  $H$  in  $L^2(\mathbb{R}^k)$  is the sequence  $\left( \prod_{j=1}^k F_{\mu_j}(\xi_j) \right)_{\mu \in \mathbb{N}^k}$ , which gives, for  $\mu, \mu' \in \mathbb{N}^k$  and  $g = (x, y, u)$ ,

$$\varphi^{\Pi_S, \mu, \mu'}(g) = \langle \Pi(g)(F_\mu), F_{\mu'} \rangle = e^{-i\frac{u}{4}} \prod_{j=1}^k \Phi_{\mu_j, \mu'_j}(x_j, y_j).$$

By (25) the function  $z \rightarrow \varphi^{\Pi_S, \mu, \mu'}(zx, zy, z^2u)$  is holomorphic on  $\mathbb{C}$ . Let

$$R_{a,\delta} = \{\alpha + i\beta \mid |\alpha| < a, |\beta| < \delta\} \subset \mathbb{C}.$$

For some constant  $C_{a,\delta}$ , and  $z \in \overline{R_{a,\delta}}$ ,

$$\left| \varphi^{\Pi_S, \mu, \mu'}(zx, zy, z^2u) \right| \leq C_{a,\delta} e^{\frac{1}{2}a\delta|u|} \prod_{j=1}^k e^{\delta^2(x_j^2 + y_j^2)}.$$

We now look for conditions on  $a, \delta$  ensuring that the right hand side lies in  $L^q(pdg)$ . We recall [Hu] that

$$p(x, y, u) = \int_{\mathbb{R}} e^{i\lambda u} Q(x, y, \lambda) d\lambda = c_k \int_{\mathbb{R}} e^{i\lambda u} \prod_{j=1}^k \frac{2\lambda}{sh 2\lambda} e^{-\frac{\lambda}{ih 2\lambda}(x_j^2 + y_j^2)} d\lambda.$$

Noting that  $Q(x, y, \lambda) = \prod_{j=1}^k Q_1(x_j, y_j, \lambda)$  is even w.r. to  $\lambda$ , we get, for  $q \geq 1$ ,

$$\frac{1}{2} \int_{\mathbb{R}} e^{\frac{q}{2}a\delta|u|} p(x, y, u) du \leq \int_{\mathbb{R}} ch\left(\frac{q}{2}a\delta u\right) p(x, y, u) du = Q(x, y, ia\delta\frac{q}{2}).$$

We need the convergence of

$$\int_{\mathbb{R}^{2k}} \prod_{j=1}^k e^{q\delta^2(x_j^2 + y_j^2)} Q_1(x_j, y_j, i\frac{q}{2}a\delta) dx dy = c \prod_{j=1}^k \int_{\mathbb{R}^2} e^{(q\delta^2 - \frac{1}{2}\frac{qa\delta}{igqa\delta})(x_j^2 + y_j^2)} dx_j dy_j,$$

which holds for  $qa\delta \leq \frac{\pi}{4}$  and  $a > 2\delta$ . Thus, taking  $a = N \in \mathbb{N}$ ,  $\varphi^{\Pi_S, \mu, \mu'}$  satisfies the assumptions of step 1 on

$$\Omega = \cup_{N \geq 2} R_{N, \frac{\pi}{4qN}}.$$

Plancherel formula for  $\mathbb{H}_k$  (see e.g. [T, Theorem 1.3.1] or [CG, p.154]) involves the representations

$$\rho_h(x, y, u) = e^{-\frac{i}{4}hu} \Pi_S(x, hy, 0).$$

By the Stone-Von Neumann theorem [T, theorem 1.2.1] every irreducible unitary representation  $\Pi$  of  $\mathbb{H}_k$  satisfying  $\Pi(0, 0, u) = e^{-\frac{i}{4}hu}$  for a real  $h \neq 0$  is unitarily equivalent to  $\rho_h$ . Hence  $\rho_{\beta^2}$  (resp.  $\rho_{-\beta^2}$ ) is unitarily equivalent to  $\Pi_S \circ \delta_\beta$ , (resp.  $\Pi_S \circ \sigma \circ \delta_\beta$ ),  $\beta > 0$ , where  $\sigma$  is the automorphism of  $\mathbb{H}_k$  defined by  $\sigma(x, y, u) = (x, -y, -u)$ .

Since  $\Pi_S(L) = \Pi_S \circ \sigma(L)$ , we get  $\varphi^{\Pi_S \circ \sigma, \mu, \mu'} = \varphi^{\Pi_S, \mu, \mu'} \circ \sigma = \overline{\varphi^{\Pi, \mu, \mu'}}$ , hence

$$\mathcal{F} = \{\varphi^{\Pi_S, \mu, \mu'} \circ \delta_\beta, \overline{\varphi^{\Pi_S, \mu, \mu'}} \circ \delta_\beta, \beta > 0, \mu, \mu' \in \mathbb{N}^k\}.$$

The conditions of step 1 are satisfied by  $\varphi^{\Pi, \mu, \mu'} \circ \delta_\beta$ , replacing  $R_{a, \delta}$  by  $R_{\beta a, \beta \delta}$ , which ends the proof of theorem 12 for  $\mathbb{H}_k$ . Taking remark 5 into account, the set  $\cup_{\mu, \mu', n} \{h_n^{\Pi_S, \mu, \mu'}, \overline{h_n^{\Pi_S, \mu, \mu'}}\}$  is total in  $L^2(\mathbb{H}_k, pdg)$ .

**Step 3.** We first recall some more facts on representations and compute the set  $\mathcal{F}$  for step 2 stratified groups. We shall follow Cygan's scheme [Cy].

Let  $l \in \mathcal{G}^*$ . Among the Lie subalgebras  $\mathcal{M} \subset \mathcal{G}$  satisfying  $\langle l, [X, Y] \rangle = 0$  for every  $X, Y \in \mathcal{M}$ , some have minimal codimension  $m_l$  and are denoted by  $\mathcal{M}_l$ . Then the map

$$Z \in \mathcal{M}_l \rightarrow e^{i\langle l, Z \rangle} \quad (26)$$

is a representation of the subgroup  $\exp \mathcal{M}_l$  and induces an irreducible unitary representation of  $G$  as follows [CG, theorems 1.3.3, 2.2.1 and p 41] : One chooses independent vectors  $(X_j)_{j=1}^{m_l}$  such that  $\mathcal{G} = \mathcal{M}_l + \text{span}\{(X_j)_{j=1}^{m_l}\}$ . For  $(g, \xi) \in G \times \mathbb{R}^{m_l}$  there exist  $(\xi', M) \in \mathbb{R}^{m_l} \times \mathcal{M}_l$  such that

$$\exp\left(\sum_{i=1}^{m_l} \xi_i X_i\right) \cdot g = \exp M \cdot \exp\left(\sum_{i=1}^m \xi'_i X_i\right).$$

Then, for  $F \in L^2(\mathbb{R}^{m_l})$ ,

$$\Pi_l(g)(F)(\xi) = e^{i\langle l, M \rangle} F(\xi'). \quad (27)$$

The set of  $\mathcal{C}^\infty$  vectors for  $\Pi_l$  is  $\mathcal{S}(\mathbb{R}^k)$  [CG, corollary 4.1.2]. Every irreducible unitary representation of  $G$  is equivalent to a representation constructed in this way; different  $\mathcal{M}_l, \mathcal{M}_{l'}$  and different  $l, l'$  in the same coadjoint orbit induce equivalent representations [CG, theorems 2.2.2, 2.2.3, 2.2.4].

By Kirillov theory there is an integer  $k$  and a set  $\mathcal{U}_0 \subset \mathcal{G}^*$  of "generic" orbits with maximal dimension  $2k$ , such that  $m_l = k$  for  $l \in \mathcal{U}_0$ . The Plancherel measure is supported by  $\mathcal{U}_0$  [CG, theorem 4.3.10].

We now compute such a  $\Pi_l$  when  $G$  is step 2. Let  $U_1, \dots, U_d$  be a basis of the central layer  $\mathcal{Z}$  and let  $\chi_1, \dots, \chi_n$  be a basis of the first layer  $V_1$  of  $\mathcal{G}$ .

Let  $l \in \mathcal{G}^*$  and let  $\lambda = \sum_{j=1}^d \lambda_j U_j^*$  be its central part, identified with a vector  $\lambda \in \mathbb{R}^d$ . Let  $A_\lambda$  be the  $n \times n$  matrix with coefficients  $\langle \lambda, [\chi_j, \chi_h] \rangle$ .

By Campbell-Hausdorff formula, for  $Y \in \mathcal{G}, X \in V_1, U \in \mathcal{Z}, g = \exp(X + U)$ ,

$$\exp \text{Adg}(Y) = g \exp Y g^{-1} = e^{[X, Y]} \exp Y = \exp(Y + [X, Y]),$$

hence the coadjoint orbit of  $l$ , i.e.  $\{l \circ \text{Adg}, g \in \mathcal{G}\} \subset \mathcal{G}^*$ , is  $l + \text{range } A_\lambda$ .

We now assume that  $l$  lies in  $\mathcal{U}_0$ , so that the range of  $A_\lambda$  has dimension  $2k$ . There exists an orthogonal matrix  $\Omega_\lambda$  such that

$$A_\lambda = \Omega_\lambda A'_\lambda \Omega_\lambda^*$$

where  $A'_\lambda$  is block diagonal, the non zero blocks having the form

$$\nu_j(\lambda) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \nu_j(\lambda) > 0. \quad (28)$$

The new basis of  $V_1$  (defined by the columns of  $\Omega_\lambda$ ) is denoted by  $X_1, Y_1, \dots, X_k, Y_k, S_1, \dots, S_{n-2k}$ , so that

$$\langle \lambda, [X_j, X_h] \rangle = 0 = \langle \lambda, [Y_j, Y_h] \rangle, \quad \langle \lambda, [X_j, Y_h] \rangle = \nu_j(\lambda) \delta_{jh}, \quad 1 \leq j, h \leq k. \quad (29)$$

We denote  $t = \Omega_\lambda(x, y, s) \in \mathbb{R}^n$ , where

$$\sum_{j=1}^n t_j \chi_j = \sum_{j=1}^k x_j X_j + y_j Y_j + \sum_{h=1}^{n-2k} s_h S_h = X + Y + S \in V_1.$$

Choosing  $\mathcal{M}_l = \mathcal{Z} + \text{span}\{Y_j, S_h\}_{1 \leq j \leq k, 1 \leq h \leq n-2k}$ , let us compute  $\Pi_l$ . By definition  $\Pi_l(\exp u_j U_j) = e^{iu_j \lambda_j}$ . For  $g = \exp(X + Z)$  and  $Z = Y + S$ ,

$$\begin{aligned} \exp\left(\sum_{j=1}^k \xi_j X_j\right) g &= \exp\left[\sum_{j=1}^k \xi_j X_j, X + Z\right] g \exp\left(\sum_{j=1}^k \xi_j X_j\right) \\ &= \exp\left(\left[\sum_{j=1}^k \xi_j X_j, X + Z\right] + \frac{1}{2}[X, Z]\right) \exp Z \exp X \exp\left(\sum_{j=1}^k \xi_j X_j\right) \\ &= \exp M \exp\left(\sum_{j=1}^k (\xi_j + x_j) X_j\right). \end{aligned}$$

Hence, by (27) and (29), for  $F \in L^2(\mathbb{R}^k)$ ,

$$\Pi_l(g)(F)(\xi) = e^{i\langle l, M \rangle} F(\xi + x) = e^{i \sum_{j=1}^k \nu_j y_j (\xi_j + \frac{1}{2} x_j)} e^{i\langle l, Y + S \rangle} F(\xi + x). \quad (30)$$

Since we may replace  $l$  by  $l'$  in the orbit of  $l$ , we may suppose  $\langle l, Y_j \rangle = 0$ ,  $1 \leq j \leq k$ . In particular, by (30),

$$\Pi_l(X_j) = \frac{\partial}{\partial \xi_j}, \Pi_l(Y_j) = i\nu_j \xi_j, \quad 1 \leq j \leq k, \Pi_l(S_h) = i\langle l, S_h \rangle I, \quad 1 \leq h \leq n-2k.$$

Since  $\Omega_\lambda$  is orthogonal,  $-L = \sum_{j=1}^k (X_j^2 + Y_j^2) + \sum_{h=1}^{n-2k} S_h^2$ , which entails

$$\Pi_l(L) = \sum_{j=1}^k -\frac{\partial^2}{\partial \xi_j^2} + \nu_j^2 \xi_j^2 + \sum_{h=1}^{n-2k} \langle l, S_h \rangle^2 I.$$

A basis of eigenvectors of  $\Pi_l(L)$  is thus  $\left( \prod_{j=1}^k F_{\mu_j}(\sqrt{\nu_j} \xi_j) \right)_{\mu \in \mathbb{N}^k}$ . By (30) and (25), for  $g = (x, y, s, u)$ ,

$$\varphi^{\Pi_l, \mu, \mu'}(g) = e^{i\langle \lambda, u \rangle} e^{i \sum_{h=1}^{n-2k} s_h \langle l, S_h \rangle} \prod_{j=1}^k \frac{1}{\sqrt{\nu_j}} \Phi_{\mu_j, \mu'_j}(\sqrt{\nu_j} x_j, \sqrt{\nu_j} y_j).$$

Hence, for  $z \in R_{a, \delta}$  and some constant  $C_{a, \delta}$ , with  $t = \Omega_\lambda(x, y, s)$ ,

$$\begin{aligned}
\left| \varphi^{\Pi_l, \mu, \mu'}(zt, z^2u) \right| &\leq C_{a, \delta} e^{2a\delta|\langle \lambda, u \rangle|} e^{\delta \sum_{h=1}^{n-2k} |s_h \langle l, S_h \rangle|} \prod_{j=1}^k \frac{1}{\sqrt{\nu_j}} e^{\delta^2 \nu_j (x_j^2 + y_j^2)} \\
&= e^{2a\delta|\langle \lambda, u \rangle|} w_{a, \delta, l}(x, y, s).
\end{aligned}$$

By [Cy, corollary 5.5] the heat kernel  $p(t, u)$  is the Fourier transform of  $CQ(t, \lambda)$  w.r. to the central variables, where

$$Q(t, \lambda) = \prod_{h=1}^{n-2k} e^{-\frac{1}{2}s_h^2} \prod_{j=1}^k Q_1(x_j, y_j, \frac{\nu_j}{4}) = Q(t, -\lambda).$$

Again, we need the convergence of

$$\int_{\mathbb{R}^n} w_{a, \delta, l}^q(x, y, s) \prod_{h=1}^{n-2k} e^{-\frac{1}{2}s_h^2} \prod_{j=1}^k Q_1(x_j, y_j, \frac{iq a \delta \nu_j}{2}) dx dy ds,$$

which holds if  $q a \delta \max \nu_j \leq \frac{\pi}{4}$  and  $a > 2\delta$ . This ends the proof of theorem 12. ■

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